Iterative Solution of the Three-Body Problem and System Simulation

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Abstract: The three-body problem is a fundamental model in astrophysics that studies the law-of-motion problem of three objects that are considered as masses under the gravitational influence. This paper utilizes classical mechanical equations and computer's advantages in terms of numerical operations to iteratively solve the three-body problem and determine its bound state. In this experiment, the three objects are analyzed by force; equations are listed using Newton's second law and law of universal gravitation; and the MATLAB program is written according to the equations for iterative solution, which calculates the position and velocity of the three objects at a specific time under different circumstances. In addition, in this paper, the Gui design and animation of MATLAB are utilized to investigate and display the problem. A simple and friendly interface is designed, the position and velocity of the objects are also shown on the axes in real time through the erasure animation, which is an attempt to perform astrophysical experiments on the computer.

1. Introduction

The three-body problem is a well-known problem in classical mechanics, quantum mechanics, and astronomy that has not yet been completely solved. In astrophysics, the three-body problem focuses on the gravitational motion of three bodies that can be viewed as three mass points under the influence of gravity on each other—i.e., given the mass, initial positions, and initial velocities of three bodies, the variation of their positions and velocities with time, macroscopic laws, and global properties of their motion can be further discussed.

Due to the constant changes of positions and velocities of the three objects in the system, the three-body problem cannot be streamlined in a similar solution to the way the two-body problem is treated. Since the three-body problem cannot be resolved strictly, in studying the motion of celestial bodies, we can only apply various approximate solutions according to the actual situation. The first one is analytical method, the basic principle of which is to expand the coordinates and velocity of celestial bodies into approximate analytical expressions in the form of series of time or other small parameters, so as to discuss the changes of coordinates or orbital elements of celestial bodies over time. The second method is qualitative research, which adopts the qualitative theory of differential equations to study the macroscopic and global laws of three-body motion over a long period of time. The third category is the numerical method, which directly applies the computational method in differential equations to measure the exact positions and velocities of the celestial bodies at certain moments. Each of these three types of methods has advantages and disadvantages.

Since the difficulties in solving three-body problem lies in its uncertainty and unpredictability, one can consider a third type of approach based on force analysis and classical mechanical equations, taking advantage of iterative computation of computers to find its numerical solution.

2. Model assumptions

(1) Ignores the relativistic effects.

(2) The cosmic space in which it is located is isotropic, regardless of the role of other celestial bodies on the system.

(3) Ignoring tidal forces, resistance of intergalactic space particles, and assuming no energy loss during motion.

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(4) Consider celestial bodies as masses, regardless of their collisions with each other.
(5) The three celestial bodies have the same mass.

3. Description of symbols

Table 1. Symbols & Descriptions

G	The gravitational constant
m	Celestial masses
t	Running time
d	Location of celestial bodies
p	Celestial velocity
a	Celestial acceleration
x,y,z	The position of the celestial body in the Cartesian coordinates system
u,v,w	The velocity of the objects in the Cartesian coordinate system
a_x,a_y,a_z	The acceleration of the objects in the Cartesian coordinate system
F_x,F_y,F_z	Inter-object forces
E_1,E_2,E_3	The difference between the kinetic and potential energy of celestial bodies

4. Model building

4.1 State of Specific Moment

Since the positions and velocities of the three objects vary from moment to moment, we can now take a short interval of time: dt. Assuming that the parameters of the system do not change throughout this time, we can consider the state of the system at each moment and estimate the state at the next moment, after dt. At each moment, each celestial body has a specific position and velocity, which means there would be six unknown quantities. By applying Newton's second law and law of universal gravitation between each two celestial bodies, we could have equation sets composed of six equations, which is sufficient enough to solve these six unknown quantities. If dt is infinitely small, a precise solution can then be found. However, this condition is not practical to be realized, so we divide a period of time into many small parts and solve iteratively, via computer programming, to obtain the changes of each parameter over time and the end state of the system.

4.1.1 Force Analysis

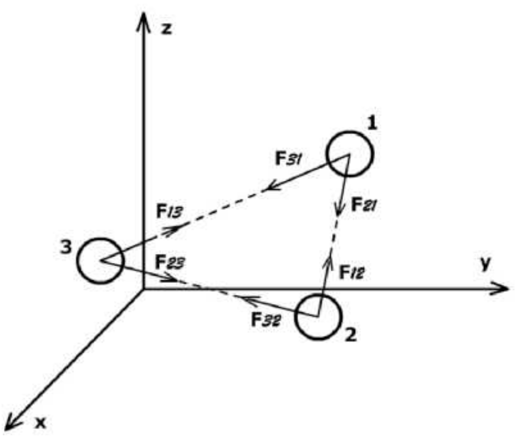

Figure 1. Force analysis of the three-body problem

As shown in Figure 1, each object is subject to the gravitational force of the other two objects. For ease of calculation, the effect of object 1 on object 2, F_{12},the effect of object 2 on object 3, F_{23}, and the effect of object 3 on object 1, F_{13}, are decomposed into F_{12x}, F_{12y}, F_{12z}, F_{23x}, F_{23y}, F_{23z}, and F_{31x}, F_{31y}, F_{31z}, respectively, in a Cartesian coordinate system. Since $F_{12} = F_{21}$, $F_{23} = F_{32}$, $F_{31} = F_{13}$, the right-hand equivalent of the equation can be used to represent the left.

Under the action of gravity F_{12}, F_{23}, and F_{31}, acceleration of celestial body 1, 2, and 3 is $\frac{F_{12}+F_{31}}{m}$, and $\frac{F_{12}+F_{23}}{m}$, $\frac{F_{23}+F_{31}}{m}$, respectively.

4.1.2 Laws of Motion

The three objects follow the following laws of motion and iterative equations.

1) Newton's Law of Gravitation

$$F_{12} = G\frac{m^2}{(d_1-d_2)^2} \tag{1}$$

$$F_{23} = G\frac{m^2}{(d_2-d_3)^2} \tag{2}$$

$$F_{31} = G\frac{m^2}{(d_3-d_1)^2} \tag{3}$$

2) Newton's Second Law

$$a_1 = G\frac{F_{12}+F_{31}}{m} \tag{4}$$

$$a_2 = G\frac{F_{12}+F_{23}}{m} \tag{5}$$

$$a_3 = G\frac{F_{23}+F_{31}}{m} \tag{6}$$

3) Speed Iterations

$$p_1 = p_1 + a_1 \times t \tag{7}$$

$$p_2 = p_2 + a_2 \times t \tag{8}$$

$$p_3 = p_3 + a_3 \times t \tag{9}$$

4) Iteration of Coordinating Values

$$d_1 = d_1 + p_1 \times t + \frac{1}{2} \times a_1 \times t^2 \tag{10}$$

$$d_2 = d_2 + p_2 \times t + \frac{1}{2} \times a_2 \times t^2 \tag{11}$$

$$d_3 = d_3 + p_3 \times t + \frac{1}{2} \times a_3 \times t^2 \tag{12}$$

The solution would also need to decompose the iterative equations of force F, acceleration a, velocity, and coordinate values by the Cartesian coordinate system as shown in appendix 1.

The position and velocity of the three objects at any given moment can be resolved by computer programming.

4.2 Judgement on Bound State

Three objects are sometimes closer to or further away from each other, and there is always interaction between them. None one of these objects escape into space and never return, which is called the bound state of three-body motion. In order to establish whether a given three-body system is a bound state, we calculate the kinetic energy and potential energy of each object in the system and compare them to obtain the result.

4.2.1 Problem Analysis and Simplification

As shown in Figure 2, in a two-body system, one of the celestial bodies is the coordinate origin to establish a coordinate system. If the other celestial body to its gravitational potential energy is less than the kinetic energy of the celestial body in this coordinate system (Equation (13)), then the celestial body will escape out and will not return. Otherwise, if the potential energy is greater than the kinetic energy (Equation (14)), the celestial body will have then been in motion around the celestial body located in the coordinate origin.

$$\frac{1}{2}m \times v^2 - G\frac{M \times m}{r} > 0 \tag{13}$$

$$\frac{1}{2}m \times v^2 - G\frac{M \times m}{r} < 0 \tag{14}$$

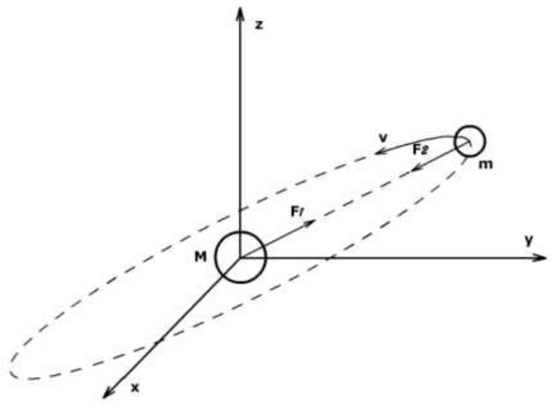

Figure 2. Two-body movements

We now examine the case of three-body. Figure 3 illustrates a simplified model in researching the kinetic energy of the celestial body 1. By positioning the center of the triangle, formed by connecting the centers of mass in each of the three bodies, to be the coordinate's origin, the estimated kinetic energy of body 1 will then be its kinetic energy in the system. Mathematical relation shows that if the velocities of body 2 and 3 are V2 and V3, the velocity of the midpoint of their line of connection would be

$$v = \frac{v_2 + v_3}{2} \tag{15}$$

Relative velocity of celestial body 1 to this point,

$$\Delta v = \frac{v_2 + v_3}{2} - v_1 \tag{16}$$

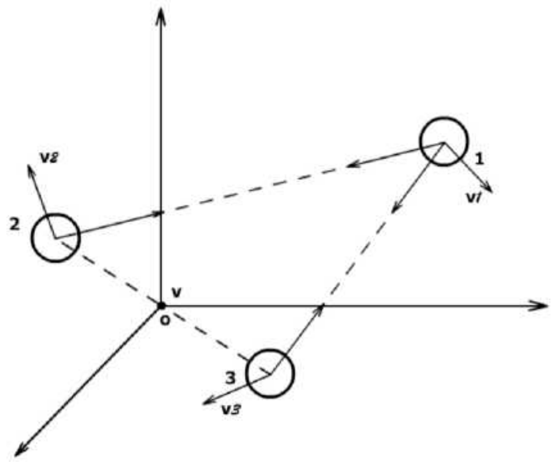

Figure 3. Three-body movements

4.2.2 Determination of Bounding State

Kinetic energy of celestial body 1,

$$V_1 = \frac{1}{2} m \times (\frac{v_2 + v_3}{2} - v_1)^2 \tag{17}$$

Potential Energy of celestial body 1,

$$H_1 = G \times \frac{m^2}{(d_2 - d_1)} + G \times \frac{m^2}{(d_1 - d_3)^2} \tag{18}$$

Thus,

$$E_1 = V_1 - H_1 = \frac{1}{2} m \times (\frac{v_2 + v_3}{2} - v_1)^2 - G \times \frac{m^2}{(d_2 - d_1)_2} - G \times \frac{m^2}{(d_1 - d_3)^2} \tag{19}$$

would be the formula for determining the bound state.

If $E_i > 0$, then it is not a bound state for object 1; otherwise, if it <0, then it is in the bound state. Similarly, there is the formula for the determination of object 2 and object 3:

$$E_2 = V_2 - H_2 = \frac{1}{2} m \times (\frac{v_1 + v_3}{2} - v_2)^2 - G \times \frac{m^2}{(d_2 - d_1)_2} - G \times \frac{m^2}{(d_3 - d_2)^2}$$

$$E_3 = V_3 - H_3 = \frac{1}{2} m \times (\frac{v_1 + v_2}{2} - v_3)^2 - G \times \frac{m^2}{(d_1 - d_3)_2} - G \times \frac{m^2}{(d_3 - d_2)^2}$$

These three formulas can also be solved by decomposing the equations in a Cartesian coordinate system, the results of which are located in appendix 2.

5. Model solving and system simulation

5.1 Programming and Interfacial Design

Utilizing the MATLAB language to program, code and comment as referred in appendix 3. The graphic interface in the program is shown below in Figure 4. By clicking on the “start” button, the popped dialog box will provide blank space to enter parameter values. The default parameters are: G (the actual gravitational constant), and m (solar mass). The position and velocity of the planet are randomly set.

Figure 4. Parameters Input

Once you click on "OK," the program will read in the parameters entered in the dialog box (otherwise it will use the default values), and start the iterative computation according to the formulas. The program will determine the position changes of the three objects at each dt step and display them instantly in the coordinate system so we can see the animation of the three objects. (The celestial bodies will move outside of the coordinate axis in some cases).

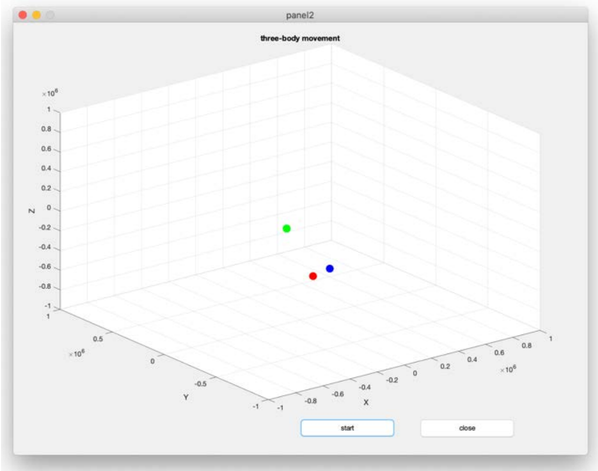

Figure 5. Animation Display

After the iteration is finished, the coordinates of the three bodies' final states (x_1, z_1 ,x_2, y_2, z_2, x_3, y_3, z_3) will be displayed on the current window. The difference between the kinetic energy of each object E_1 ,E_2, E_3 is calculated automatically. If the three bodies' kinetic energies are smaller than the potential ones, the interface will pop up the following message, saying: "It is an engage state, which means that the body will come back if time is long enough.” Otherwise, it will show “it is not an engage state, which means that the body will not come back no matter how much time is spent”.

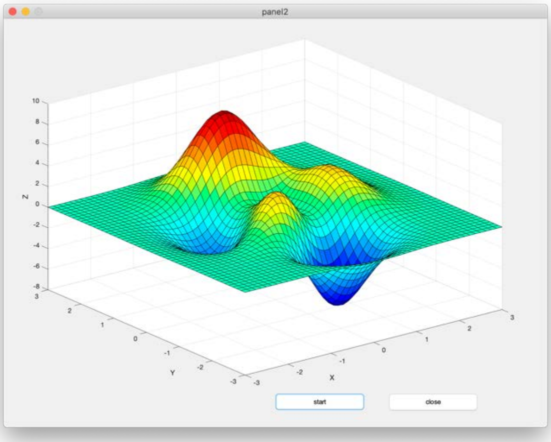

Figure 6. Result Obtained

Displays the solution results in the Command Window:

$$x_1 = -3.0776e + 005$$

$$y_1 = -7.0934e + 005$$

$$z_1 = -9.6242e + 005$$

$$x_2 = -3.3810e + 005$$

$$y_2 = -1.1578e + 005$$

$$z_2 = -6.0977e + 003$$

$$x_3 = 9.2697e + 004$$

$$y_3 = -5.9355e + 005$$

$$z_3 = -9.5631e + 005$$

$$E_1 = 1.5285e + 040$$

$$E_2 = 4.8756e + 040$$

$$E_3 = 1.6720e + 040$$

By changing the initial values of locations and velocities, there could be various solutions for determining if the object is in a bound state. One should note that if the coordinates of two bodies on either axis are the same, that could result in zero becoming a denominator, which would give wrong output.

5.2 Model Testing

By changing each parameter in the dialog box and running the program, we could discover that the objects move faster when G and m are relatively larger, as the distance is getting larger. Accordingly, when G and m are relatively smaller, the movement of the body slows down. If we set the initial velocity of one of these objects to 0, the body will start moving at a lower velocity. All of this is in line with reality, which makes the model practically applicable.

If all m are set to a smaller order of magnitude, such as 1.989 x 1019, where the initial velocities of all three objects are set to 0, run the program and it will show the body being almost inactive. The current window will then return the following values:

$$x_1 = 5.0006, y_1 = 5.9817, z_1 = 7.0016$$

$$x_2 = 25.0000,\ y_2 = 9.9970,\ z_2 = 14.9996$$

$$x_3 = 10.0008,\ y_3 = 6.9847,\ z_3 = 4.0021$$

$$E_1 = -5.5053e + 015,\ E_2 = -2.2982e + 015,\ E_3 = -5.7897e + 015$$

The dialog box shows that “It is an engage state, which means that the body will come back if time is long enough”, meaning this is a bound state. A bound state would also occur if the initial velocity of the body is set to be a very small order of magnitude.

6. Model evaluation and outlook

This paper combines quantitative numerical computation and qualitative animation representation to solve the three-body problem. By setting up parameters in a dialog box and modifying the iteration step, axis range, and animation playback frequency in the source program, we could theoretically solve the three-body problem, in any case, with simple and direct visualization. Of course, this paper also contains the following shortcomings and deficiencies in the development of the relevant models

1) The calculation results are found to be greatly affected by the iterative step length in the simulation process. Changing the step length to 0.001 in the source program shows that the trajectory of the ball is completely different from when the step length was 0.01. It can be deduced that the iterative step length greatly affects the accuracy of the operation.

2) The simulation of the actual situation is not comprehensive enough, such as the spatial scale of the universe, the distance between objects, the speed of travel, etc., and more data needs to be found to improve it.

At the same time, there is still room for developing the models in this paper, such as the effects of introducing relativistic effects and the use of calculus to calculate gravity when the distance between objects is trivial enough. In addition, if the computer processing speed is fast enough, it is possible to set a tiny iteration step and a high screen refresh frequency, so that the results will be closer to perfect.

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[10] Appendix

[11] Appendix 1

[12] Newton's Law of Universal Gravitation:

$$[13]\ F_{x12} = G\frac{m^2}{(x_1 - x_2)^2}$$

$$[14]\ F_{y12} = G\frac{m^2}{(y_1 - y_2)^2}$$

$$[15]\ F_{z12} = G\frac{m^2}{(z_1 - z_2)^2}$$

$$[16]\ F_{x23} = G\frac{m^2}{(x_2 - x_3)^2}$$

$$[17]\ F_{y23} = G\frac{m^2}{(y_2 - y_3)^2}$$

$$[18]\ F_{z23} = G\frac{m^2}{(z_2 - z_3)^2}$$

$$[19]\ F_{x31} = G\frac{m^2}{(x_3 - x_1)^2}$$

$$[20]\ F_{y31} = G\frac{m^2}{(y_3 - y_1)^2}$$

$$[21]\ F_{z31} = G\frac{m^2}{(z_3 - z_1)^2}$$

[22] Newton's Second Law:

$$[23]\ a_{x1} = \frac{F_{12} + F_{31}}{m}$$

$$[24]\ a_{y1} = \frac{F_{12} + F_{31}}{m}$$

$$[25]\ a_{z1} = \frac{F_{12} + F_{31}}{m}$$

$$[26]\ a_{x2} = \frac{F_{12} + F_{23}}{m}$$

$$[27]\ a_{y2} = \frac{F_{12} + F_{23}}{m}$$

$$[28]\ a_{z2} = \frac{F_{12} + F_{23}}{m}$$

$$[29]\ a_{x3} = \frac{F_{23} + F_{31}}{m}$$

$$[30]\ a_{y3} = \frac{F_{23} + F_{31}}{m}$$

$$[31]\ a_{z3} = \frac{F_{23} + F_{31}}{m}$$

[32] Velocity Iteration:

$$[33]\ u_1 = u_1 + a_{x1} \times t$$

$$[34]\ v_1 = v_1 + a_{y1} \times t$$

$$[35]\ w_1 = w_1 + a_{z1} \times t$$

$$[36]\ u_2 = u_2 + a_{x2} \times t$$

$$[37]\ v_2 = v_2 + a_{y2} \times t$$

$$[38]\ w_2 = w_2 + a_{z2} \times t$$

$$[39]\ u_3 = u_3 + a_{x3} \times t$$

$$[40]\ v_3 = v_3 + a_{y3} \times t$$

$$[41]\ w_3 = w_3 + a_{z3} \times t$$

[42] Coordinates Values Iteration:

$$[43]\ x_1 = x_1 + u_1 \times t + \frac{1}{2} \times a_{x1} \times t^2$$

$$[44]\ y_1 = y_1 + v_1 \times t + \frac{1}{2} \times a_{y1} \times t^2$$

$$[45]\ z_1 = z_1 + w_1 \times t + \frac{1}{2} \times a_{z1} \times t^2$$

$$[46]\ x_2 = x_2 + u_2 \times t + \frac{1}{2} \times a_{x2} \times t^2$$

$$[47]\ y_2 = y_2 + v_2 \times t + \frac{1}{2} \times a_{y2} \times t^2$$

$$[48]\ z_2 = z_2 + w_2 \times t + \frac{1}{2} \times a_{z2} \times t^2$$

$$[49]\ x_3 = x_3 + u_3 \times t + \frac{1}{2} \times a_{x3} \times t^2$$

$$[50]\ y_3 = y_3 + v_3 \times t + \frac{1}{2} \times a_{y3} \times t^2$$

$$[51]\ z_3 = z_3 + w_3 \times t + \frac{1}{2} \times a_{z3} \times t^2$$

[52] Appendix2

$$E_1 = \frac{1}{2}m \times \left[(\frac{u_2 + u_3}{2} - u_1)^2 + (\frac{v_2 + v_3}{2} - v_1)^2 + (\frac{w_2 + w_3}{2} - w_1)^2 \right]$$

[53] $$- G \times \frac{m^2}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}} - G \times$$

$$\frac{m^2}{\sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2}}$$

$$E_2 = \frac{1}{2}m \times \left[(\frac{u_1 + u_3}{2} - u_2)^2 + (\frac{v_1 + v_3}{2} - v_2)^2 + (\frac{w_1 + w_3}{2} - w_2)^2 \right]$$

[54] $$- G \times \frac{m^2}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}} - G \times$$

$$\frac{m^2}{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2}}$$

$$E_3 = \frac{1}{2}m \times \left[(\frac{u_1 + u_2}{2} - u_3)^2 + (\frac{v_1 + v_2}{2} - v_3)^2 + (\frac{w_1 + w_2}{2} - w_3)^2 \right]$$

[55] $$- G \times \frac{m^2}{\sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2}} - G \times$$

$$\frac{m^2}{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2}}$$